The Role of Porosity in Filtration

Numerical Methods for Constant Rate and Constant Pressure Filtration Based on Kozeny's Law

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Based upon the Kozeny law relating the rate of flow to the porosity, a method has been developed for determining (a) the pressure drop vs. the depth and the flow rate vs. the applied pressure in a fixed bed of solids, (b) the pressure vs. time relationship in constant rate filtration, and (c) the volume vs. time relationship in *constant pressure filtration*. *It is assumed that the porosity is solely a function of the pressure on the solids, thus eliminating solids such as certain naturally occurring clays in which application of pressure causes only a slow approach to the equilibrium porosity.*

Experimental porosity vs. pressure data are presented for such materials as kaolin, calcium carbonate, carbon black (*suchar*), *diatomaceous earth (hyflo)*, asbestos, and mixtures of these materials. As a satisfactory approximation, the per cent voids can be related to the pressure by a power function with powers ranging from 0.01 to 0.05 for applied pressures up to 100 lbs/sq.in.

Numerous authors (*1, 4, 6, 7, 11*) have recognized the importance of porosity in filtration. As long ago as 1937, Tour (*11*) developed equations relating the filtration resistance to the porosity variation throughout a fixed bed of solids. Experiments which he performed clearly indicated that the pressure gradient on a liquid passing through a solid was changing and that the porosity was not constant. While Tour's procedure was sound, some of his equations relating porosity to pressure were not tested, and the work was not published. Ruth (*9*), in the period 1932-37, also developed a procedure for relating the filtration variables in a solid of variable porosity. Carman (*1*) focused attention on Kozeny's equation involving the rate of flow as a function of the porosity and other variables. Several other investigators considered Kozeny's equation but have not used it in the best manner in filtration equations. In this paper the Kozeny equation is used as a basis for developing the differential equations of filtration. These equations are used in connection with fixed beds of solids, constant rate filtration of a solid with and without precoat, and constant pressure filtration with and without a precoat.

Factors Involved in Filtration

In the filtering of solids from suspensions or in any process in which a liquid flows through the interstices of a solid, the flow is in the direction of decreasing pressure. The solids are supported on a screen, cloth, or other solid bed known as the septum or *filter medium*. In ordinary filtration practice, the solids closest to the septum are packed more densely than the rest of the cake. The porosity is a minimum at the point of contact between the cake and the septum and is a maximum at the surface where the liquid enters. Since the filtration resistance depends upon the porosity, the resistance steadily increases as the liquid passes through the cake if the solids are compressible. In order to treat quantitatively the filtration problem, it is necessary to have relationships between the rate of flow, pressure on the liquid, and the porosity in addition to the various parameters. The information necessary to an understanding of the methods used in this paper is summarized as follows:

1. Relation of porosity to pressure on the solids. Data (*7, 8, 12*) are presented in addition to new data. It is indicated that the porosity may be empirically related to the pressure on the solids by a power function, in the pressure range up to 100 lb./sq.in.

2. Relation of pressure on solids to pressure on liquid. *From the standpoint of a problem in statics*, a free body analysis of the solid and liquid shows that the pressure on the solid increases as the liquid passes frictionally through the pores.

3. The Kozeny equation as presented by Carman is used as a basis for relating the rate of flow to the pressure gradient on the liquid, porosity, specific surface of the solid, and the viscosity. Other laws relating the flow rate to the pressure gradient could be used equally well.

Porosity vs. Pressure on Solids

In the field of soil mechanics, the per cent voids or void ratio is frequently determined as a function of the applied load. The apparatus employed consists essentially of a container, termed a consolidometer, in which samples can be placed and subjected to vertical loading. In the simplest type of testing, the sample is placed between porous plates and kept saturated with water while the load is applied. At the instant the load is added, the stress is applied to both the solid particles and the liquid within the interstices. Since the liquid is free to flow through and around the porous plates, the excess of liquid pressure above atmospheric, referred to as the neutral stress, causes the moisture to leave the voids (Fig. 1), resulting in a consolidation of the solids. As the water flows out, the neutral stress drops, the solids finally carrying the applied load. When the entire load is borne by the solids as indicated in Figure 2 the actual pressure on the solids is given by

$$\text{pressure on solids} = \text{applied pressure}/(1-\epsilon) \qquad (1)$$

where ϵ is the equilibrium value of the per cent voids by volume or porosity and $(1-\epsilon)$ is equal to the average true area of the solids per square foot.

Frank M. Tiller, who had received the A.I.Ch.E. Junior Member Award in 1950, was again a winner at the 1952 Cleveland meeting, where this paper earned the best presented paper award. A B.Ch.E. of the University of Louisville and a Ph.D. of the University of Cincinnati, he taught two years at Cincinnati, nine years at Vanderbilt University, and a short time at the Instituto de Oleos in Rio de Janeiro. He is now director of engineering of Lamar State College of Technology, Beaumont, Tex.

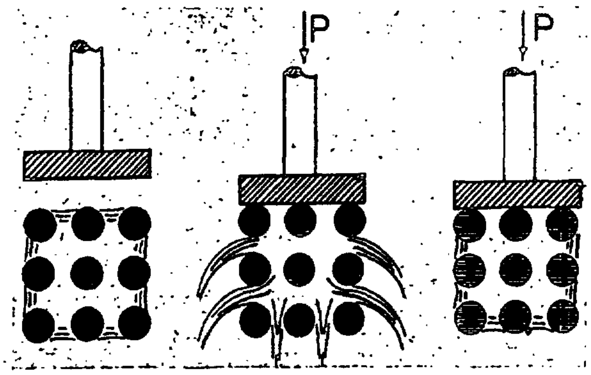


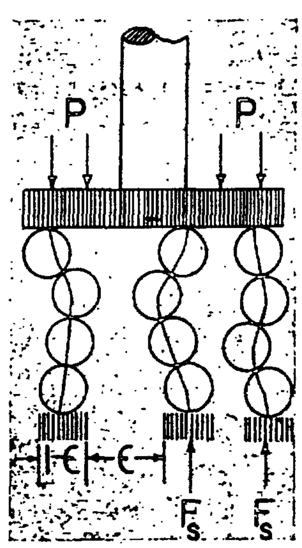
Fig. 1. Compression process in a consolidometer.

In this type of test, the liquid is not confined, and the neutral stress must approach zero. The rate of flow of the water from the sample depends upon the permeability, and the time required to reach equilibrium ranges from a few minutes for the materials used in this investigation to 24 hr. for soils classified as clays. In addition to the primary equilibrium which is reached, there is a secondary creep effect which may continue more or less indefinitely.

In another procedure known as triaxial compression testing (*10*), the sample is placed in a water-tight covering in which the liquid pressure P_w, or neutral stress, can be controlled. After adjusting the neutral stress, an independent external load is added which results in compression of the solids as indicated in Figure 3. The neutral stress acts equally in all directions and does not affect the porosity; thus the external load over and above the neutral stress $(P - P_s)$ is responsible for compression and decrease of the voids. Actually in triaxial testing, the solid is not confined laterally and changes in dimensions occur along all three axes.

In filtration, the neutral stress in the consolidometer and triaxial soil compression tests is equivalent to the liquid pressure in the interstices of the cake. In the static tests the neutral stress is constant throughout the solid; whereas in filtration the liquid pressure decreases in the direction of the flow of the filtrate. *In this investigation it is assumed that* the liquid pressure does not affect the porosity. Then the results obtained in the simple consolidometer can be used for predicting relationships between porosity and pressure on the solids in a filter cake.

Ruth (*8*) developed an ingenious permeability-compression apparatus in which filtrations could be carried out while the pressure on the solids was controlled independently. Loads were added in a manner similar to that used in the consolidometer except that a perforated plate confined the solid at both top and bottom. After loading the solids, liquid was introduced separately and allowed to flow through the solids under relatively small heads. With this procedure the porosity of the solids could be controlled at an essentially constant value while the filtration was performed.



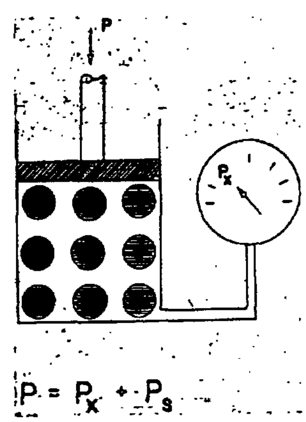


Fig. 2. Left, Equilibrium state in compression process.

Fig. 3. Above, Distribution of load between solids and liquid.

EXPERIMENTAL DETERMINATION OF POROSITY VS. APPLIED LOAD

The porosity was determined in a consolidometer consisting of a brass cup in which a wet solid could be compressed by a suitable loading procedure. The cup was approximately 5 sq.in. in area, and the solid depth was initially about 1 in. The wet solid rested on a porous plate through which moisture could flow as it was squeezed out of the solid. In most of the runs a lever system was used so that the load on the solid was four times the weight employed. An Ames dial served for measuring the thickness of the solid. The procedure consisted of the following:

1. The total depth was found before the sample was placed in the consolidometer on top of a porous stone.
2. After the wet solid was placed in the cup, a second porous stone and brass plate were placed on top of the solid and 10 min. allowed to elapse before the first reading was recorded.
3. Weights were added and 10 min. allowed between readings.
4. After this procedure, the moisture content of the solid was determined, and porosities were calculated based upon the readings and the final porosity.

The final per cent voids were calculated from the per cent moisture s of the cake at the end of the tests in the following manner:

$$\epsilon = \frac{\text{void vol.}}{\text{void vol.} + \text{solid vol.}}$$

$$= \frac{s/62.4}{s/62.4 + (1 - s)/\rho_s} \qquad (2)$$

$$\epsilon = \frac{s(\text{sp.gr.})}{s(\text{sp.gr.}) + (1 - s)} \qquad (3)$$

where ρ_s is the density of the solids and

$\rho_s/62.4$ is the sp.gr. Data and sample calculations are shown in Table 1.

The procedure outlined was applied to a diatomaceous earth (hyflo), kaolin, activated carbon (suchar), calcium carbonate, mixtures of kaolin and hyflo, mixtures of suchar and hyflo, and asbestos.

The final moisture content was ≤36%. The final porosity is calculated from Equation (2) using a value of 2.60 for the specific gravity of kaolin. Thus

$$\epsilon = \frac{(2.6)(0.2586)}{(2.6)(0.2586) - 0.7414} = 0.476$$

Calculation of porosity at 11:30

Reduction in thickness = 0.4542 − 0.3192 = 0.1350 in.

Cake thickness = initial value − 0.1350 = 1.018 − 0.135 = 0.883 in.

Cake volume = (area) (thickness) (16.4 cc./cu.in.) = vol. in cc. = (4.988) (16.4) (thickness) = (81.7) (0.883) = 72.1 cc.

Total volume of solids remains constant and can be obtained from the final porosity.

Volume of solids = (1 − .476) (62.7) = 32.8 cc.

Porosity at 11:30 = (72.1 − 32.8)/72.1 = 0.545

Empirical equation as found from Figure 5

$$\epsilon = 0.59 p_s^{-0.043}$$

In the first stages of the investigation, the effect of both time and pressure on the porosity were determined. In Figure 4 the porosity of crushed limestone is plotted against the time at different applied pressures. It can be seen that about 5 min. were required to approach apparent equilibrium closely. With natural clays and similar materials, the time required to reach an apparent equilibrium is much longer.

In some runs thickness and porosity determinations were made as the pressure was reduced, i.e., after completing the addition of loads on the sample in the consolidometer, the loads were removed and Ames dial readings recorded. Little increase in porosity was noted, and the conclusion was reached that the compression process was essentially irreversible. Carman (1) also indicated that his data pointed to the irreversibility of the compression process with most solids.

In Figure 5 some of the porosity vs. pressure data are plotted logarithmically. As a fair approximation in the ranges investigated, Figure 5 indicates that a power function of the form

$$\epsilon = \epsilon_0 p_s^{-\alpha} \quad (4)$$

Table 1.—Data and Sample Calculations for Consolidometer Tests

Porosity run No. 10 — Area of cake 4.988 sq.in.

Substance: kaolin

Final moisture content 25.86%

TIME	AMES DIAL READING IN.	APPLIED LB./SQ.IN.	CAKE THICKNESS IN.	VOL. CC.	POROSITY
Initial	0.4542	.....	1.018	83.2	.698
10:30	0.3240	.....			
10:40	0.3089	0.28	.....	....	
11:00	0.3790	1.42	0.943	77.0	.574
11:10	0.3490	3.04	0.913	74.6	.560
11:20	0.3283	5.18	0.892	72.8	.549
11:30	0.3192	6.87	0.883	72.1	.545
11:40	0.3097	8.58	0.873	71.3	.540
11:50	0.2942	12.92	0.857	70.0	.531
12:00	0.2762	19.94	0.839	68.5	.521
12:10	0.2639	27.00	0.827	67.5	.514
12:20	0.2557	33.99	0.819	66.9	.509
12:30	0.2394	47.93	0.803	65.6	.499
12:40	0.2279	61.88	0.791	64.8	.493
12:50	0.2191	75.82	0.782	63.8	.486
1:00	0.2118	89.72	0.775	63.3	.481
1:10	0.2050	103.57	0.768	62.7	.476

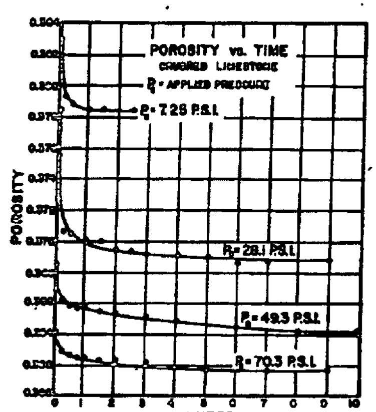


Fig. 4. Variation of porosity with time.

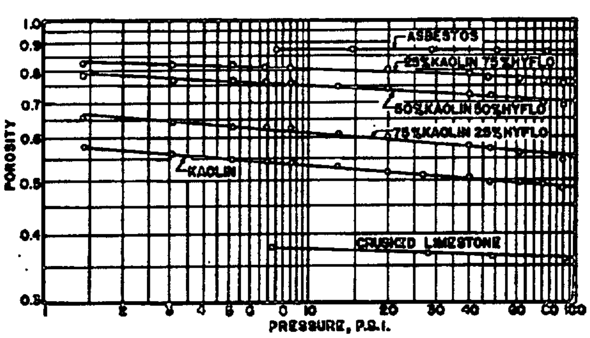


Fig. 5. Porosity vs. pressure.

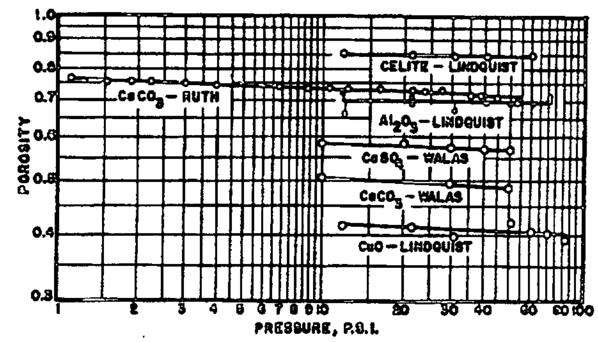


Fig. 6. Porosity vs. pressure.

Table 2.

SUBSTANCE	Literature Cited	ϵ_0	c	MAXIMUM DEVIATION IN ϵ
Crushed limestone		0.39	0.015	0.005
Calcium carbonate	Ruth (8)	0.77	0.034	0.002
Calcium carbonate	Walas (12)	0.55	0.036	0.005
Calcium sulfate	Walas (12)	0.64	0.018	0.002
Ferric oxide	Walas (12)	0.81	0.037	0.02
Magnesium carbonate	Walas (12)	0.90	0.011	0.003
100% hyflo 0% kaolin		0.88	0.014	0.01
75% hyflo 25% kaolin		0.85	0.020	0.02
50% hyflo 50% kaolin		0.81	0.032	0.02
25% hyflo 75% kaolin		0.68	0.044	0.018
0% hyflo 100% kaolin		0.59	0.045	0.008
66.7% hyflo 33.3% suchar		0.88	0.013	0.008
0% hyflo 100% suchar		0.89	0.0091	0.002
Asbestos		0.91	0.017	0.008
Aluminum oxide	Lindquist and Youle (7)	0.70	0.0047	0.03
Celite	Lindquist and Youle (7)	0.86	0.013	0.03
Copper oxide	Lindquist and Youle (7)	0.42	0.021	0.04
Carbon, norit A	Lindquist and Youle (7)	0.70	0.021	0.05

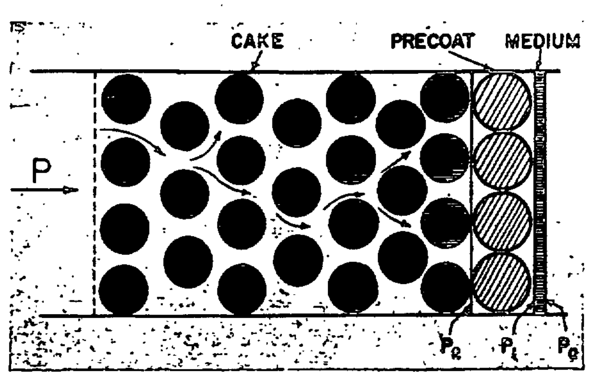


Fig. 7. Illustration of compressibility effects in a filter cake.

may be employed for representing th data where p_s is in pounds force/sq.in. In Figure 6 data of Lindquist and You (7), Ruth (8), and Walas (12) plotte in a similar manner appear to follo the same relation. Values of ϵ_0 and as found from the graphs, are given Table 2.

Terzaghi and Peck (10) have used semilogarithmic plot for representing linear relation of void ratio to applie pressure in the range of 10-100 kg. sq.cm. (142-1420 lb./sq.in.) for sand. the higher pressure, the sand grair begin to crush. For some clays the sem logarithmic relation yielded linear rel tions up to applied pressures as high 1000 kg./sq.cm. In terms of the var iables used in this investigation, th Terzaghi relation is given by

$$\frac{\epsilon}{1-\epsilon} = c_1 \ln P_s + c_2 \qquad (5$$

While this expression is adequate in th high pressure range, it is not satisfac tory at low pressures which are impor tant in filtration. In this paper the ana lytical form of the pressure-porosit relationship is not of importance excep as it may be used for interpolation. How ever, in the succeeding paper (Part II where an analytical equation is deve oped for the filtration variables, the forr of the equation is vital to the mathe matical work.

Grace (5) has indicated that th power function approximation (4) fo the porosity is not satisfactory above 10 lb./sq.in. in accord with Terzaghi an Peck. Therefore, at all times the us of Equation (4) should be examine with care.

Equation (4) is not satisfactory as p (lb./sq.in.) approaches zero (atmos pheric pressure) since an infinite valu of ϵ is predicted. Actually, after th pressure is dropped below about 0.1 lb./ sq.in. gauge pressure, the porosity tends to approach a limiting value. As a gen eral guide, Equation (4) should not be used below 0.1 lb./sq.in. for compres sible materials. Little error will be in volved if the porosities below approxi mately 0.1 lb./sq.in. are assumed con stant.

Pressures on Liquid and Solids During Filtration

A schematic diagram of the flow of liquid through a bed formed from a com pressible material is illustrated in Figure 7. The particles are shown in a manne which indicates that the porosity is de creased as the liquid passes through the solids and approaches the septum ir accord with common knowledge. In Figure 8 the liquid is shown as it passe frictionally along the particles in the filter bed. The drag on each particl

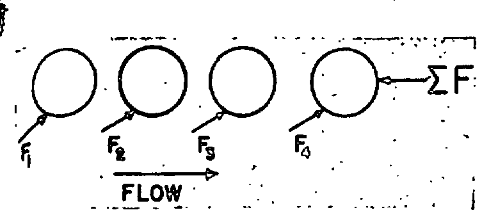


Fig. 8. Accumulative drag in a filter cake.

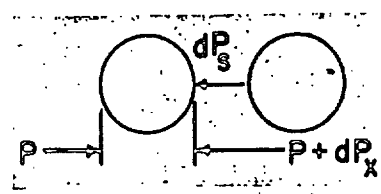


Fig. 9. Force balance around a particle.

adds to the drag on the previous particle; and consequently, the net compressive pressure builds up as the filter medium is approached, thus accounting for the decreased porosity.

At the entrance to the cake, both the pressure on the liquid P_w and the pressure on the solids are equal to the applied pressure P.† However, the applied pressure does not represent the net compressive pressure on the solids. Grace (5) and Ruth (9) have assumed that there is point contact between the particles as indicated in Figure 9 which illustrates the conditions at the surface. The pressure P at the surface of the solids is counterbalanced by a pressure $P + dP_x$ and the net increase in compressive pressure on the first layer of solids is $(-dP_x)$. Thus, as the differential pressure drop is negligible for the surface layer of the cake, the net compressive pressure P_s at the surface is essentially zero regardless of the applied pressure. The build-up of compressive pressure throughout the cake will be in accord with the relation

$$dP_s = -dP_w \qquad (6)$$

which is equivalent to

$$P_s + P_w = P \qquad (7)$$

Equation (6) is predicated on the basis of point contact and would have to be modified if there were area contact between particles. Equation (7) is equivalent to writing a force balance stating

force at surface = force on solids + force on liquid

It should be noted that pressures of Equation (7) are computed on the basis of the total cross section and not upon the area of the voids or solids.

When the liquid enters the cake $P = P_w$ and from (7) it can be seen that $P_s = 0$. As P_w decreases, P_s increases; and if the liquid leaves at atmospheric or zero gauge pressure, P_s will increase to P.

† Capital letters P, P_w, and P_s will be used for pressures expressed in lb./sq.ft., and small letters p, p_w, and p_s will be used when lb./sq.in. is employed.

The pressure on the solids can be eliminated from (7) to give a direct relationship between the liquid pressure and the porosity. The elimination of P_s can be accomplished either with the experimental porosity vs. pressure data or by means of Equation (4). If Equation (4) is used, there results (in terms of lb./sq.in.)

$$p_w = p - p_s = p - \left[\frac{\epsilon_o}{\epsilon}\right]^{1/c} \qquad (8)$$

or solving for ϵ

$$\epsilon = \epsilon_o(p - p_w)^{-c} \qquad (9)$$

The slope of the ϵ vs. p_w curve is given by

$$\frac{d\epsilon}{dp_w} = -c\epsilon_o(p - p_w)^{-c-1}$$

$$= -\frac{c}{\epsilon_o^{1/c}}\epsilon^{\frac{c+1}{c}} \qquad (10)$$

The variation of ϵ with p_w throughout a cake for various filtration pressures for the kaolin of Table 1 is illustrated in Figure 10. Equation (10) indicates that the slope of the curves depends only on the porosity and not p_w, and this is borne out by the curves in Figure 10. Even at the lowest pressure, the porosity

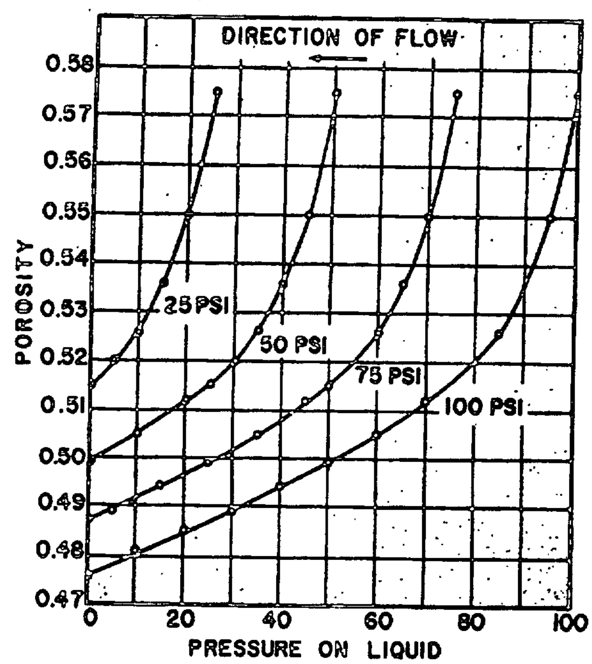


Fig. 10. Variation of porosity in a filter cake.

varies from 51.5 to 57.5%, which according to Kozeny's law would give a twofold change in filtration resistance. Several authors (*1, 6, 12*) have determined porosities under flow conditions without investigating the variations in voids throughout the bed. Since the porosity does vary, sampling procedures should be such that a true average value of moisture content is determined. At low pressures the porosity may change quite rapidly with materials like asbestos and kaolin, and low heads are no guarantee of a negligible variation in porosity.

Kozeny Equation

The Kozeny equation (*3*) relating rate of flow through a bed of solids can be derived from the following modified form of the Poiseuille equation.

$$g_c \frac{dP_x}{dx} = -\frac{k\mu}{R_H^2} u \qquad (11)$$

where k is a dimensionless constant, u the lineal velocity in ft./sec., μ the viscosity in lb.m./(ft.)(sec.); R_H the hydraulic radius, P_s the pressure in lb.f./sq.ft., $g_c P_s$ is the pressure in lbl./sq.ft., and x is the distance. The hydraulic radius is given by

$$R_H = \left(\frac{\text{flow area}}{\text{wetted perimeter}}\right)\left(\frac{\text{length of path}}{\text{length of path}}\right)$$

$$= \frac{\text{void volume}}{\text{surface of solids}} \qquad (12)$$

Since the value $\epsilon/(1-\epsilon)$ represents the ratio of the void volume to the volume of the solids, the void volume is given by

$$\text{void volume} = \frac{\epsilon}{1-\epsilon}(\text{volume of solids}) \qquad (13)$$

Substituting (13) in (12) yields

$$R_H = \frac{\epsilon(\text{volume of solids})}{(1-\epsilon)(\text{surface area of solids})} \qquad (14)$$

The surface area of the solids per cubic foot of solids (exclusive of voids) is termed the specific surface, S_o which can be substituted in Equation (14) to yield

$$R_H = \frac{\epsilon}{1-\epsilon}\frac{1}{S_o} \qquad (15)$$

The true velocity u in the interstices of the solids is given by the superficial velocity q cu.ft./(sec.)(sq.ft.) or ft./sec. divided by the per cent voids, i.e., $u = q/\epsilon$. Making substitutions in the modified Poiseuille equation leads to

$$\frac{dP_s}{dx} = -\frac{k}{g_c}\mu\frac{(1-\epsilon)^2 S_o^2}{\epsilon^3} q \qquad (16)$$

If consistent units, i.e., pound mass, poundal, foot, second; or slug, pound force, foot, second, are used, k has the value of $5g_c$. However, if inconsistent units are used with the pressure being pounds force/sq.ft. and μ being lb.m./(ft.)(sec.), k has approximately the value of 5.0. In this paper, P_s is used in lb.f./sq.ft. and k is taken as 5.0. Carman investigated Equation (16) for values of S_o ranging from 1,800 to 600,000 sq.ft./cu.ft. of solid.

For round or square particles and for cylindrical particles with length equal to diameter, S_o has the value of $6/D$. For cylindrical particles with length twice the diameter, $S_o = 5/D$. For irregular particles, Dallavalle (*2*) reports that in general the surface area is given by a constant divided by an average diameter. For rounded particles the constant was 6.1; for subrounded particles, 6.4; for sharp particles, 7.0; and for angular particles, 7.7.

Ruth (*6, 7*) has indicated that the Kozeny equation should be modified by replacing ϵ by $(\epsilon - \epsilon_o)$ where ϵ_o is a constant representing a portion of the voids unavailable for flow, termed the dead volume. Grace (*5*) has indicated that Ruth's dead volume hypothesis is not valid and that in general Kozeny's equation is inadequate. Grace also has shown that the value of k in Equation (16) is not constant and has investigated its variation with porosity. No attempt has been made in this paper to evaluate either the validity of Ruth's modification or Grace's experimental data, and calculations have been made directly with the unmodified Kozeny equation. The methods fundamentally would be the same regardless of the functional nature of the right-hand side of Equation (16).

Flow Through a Fixed Bed

The fundamental problem involved in the flow of a liquid through a fixed bed of solids revolves about the calculation of the pressure as a function of the distance. The use of Equations (*7*) and (16) in conjunction with either Equation (4) or actual porosity vs. solid pressure data permits the calculation of P_s vs. x. Variables involved may be summarized as follows:

EQUATION DESCRIPTION	VARIABLES	CONSTANTS	EQ. NO
Kozeny flow rate	P_s, ϵ, x	q, k, μ, S_o	(1
Solid and liquid pressure	P_s, P_l	P	
Porosity vs. solid pressure	ϵ, P_s	ϵ_o, ϵ	

There are four variables P_s, P_l, ϵ, and x from which it is desired to eliminate P_l and ϵ. Theoretically, P_l should be eliminated between (4) and (*7*) to give a relation between ϵ and P_s similar to (9). The porosity should then be eliminated between (9) and (16) to give a differential equation in P_s and x. Methods of this paper will be devoted to th graphical integrations of the equation.

In assuming Equation (4) to be vali the assumption is made that the porosit depends only on the pressure on th solids. This is equivalent to assumin that the equilibrium porosity is attaine immediately and is contrary to the fac as illustrated in Figure 4. When th pressure is built up slowly as in indu trial constant rate filtrations, it is prol able that the time effect on consolidatio is unimportant. However, if the rate quite high as exemplified by many labo atory investigations, the time effect prol ably cannot be neglected. In consta pressure filtration, the initial depositic of solids at a high rate may represe a case in which the time effect is impo tant. Both the nature of the solids ar the total filtration time will determi the importance of the transitory peric in which the porosity is a function time as well as pressure.

Suppose a liquid flows into a rectan ular bed of fixed thickness (Fig. 7 under an applied pressure P, while at th opposite side the liquid leaves at pressu P_1 which is necessary to overcome th resistance of the septum. At the poi where $x = 0$, the liquid pressure is P and at the other side where $x = L$, th value of P_s is P_1.

The Kozeny equation may be rea ranged for integration to give

$$\frac{kq\mu S_o^2}{g_c}\int_0^L dx = -\int_P^{P_1} \frac{\epsilon^3}{(1-\epsilon)^2} d \qquad (1$$

Reversing the limits on the right-har side to eliminate the negative sign ar integrating the left-hand side yields

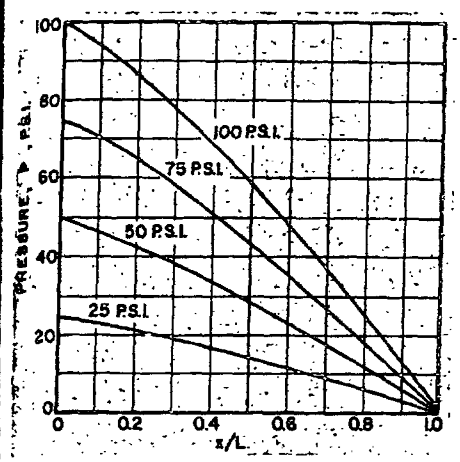
$$\frac{kq\mu S_o^2 L}{g_c} = \int_{P_1}^{P} \frac{\epsilon^3}{(1-\epsilon)^2} dP_s \qquad (18$$

where the right-hand side must b evaluated numerically. The value P represents the pressure at which th liquid leaves the solid at $x = L$, and co responds to the pressure required overcome the resistance of the septu If the upper limits in (*17*) are x and P corresponding to some arbitrary point the bed of solids, integration produces

$$\frac{kq\mu S_o^2 x}{g_c} = \int_{P_x}^{P} \frac{\epsilon^3}{(1-\epsilon)^2} dP_s \qquad (1$$

Table 3

(pound per square inch)	0	10	20	30	40	50	60	70	80	90	100
(pound per square inch)	100	90	80	70	60	50	40	30	20	10	0
ϵ	0.575	0.536	0.520	0.512	0.505	0.499	0.494	0.489	0.485	0.481	0.476
$y = \epsilon^3/(1-\epsilon)^2$	1.052	0.715	0.610	0.564	0.526	0.495	0.471	0.448	0.430	0.413	0.393
$\int^{100} y dp_s$	0	8.31	14.85	20.72	26.17	31.27	36.10	40.69	45.08	49.30	53.33
	0	0.156	0.278	0.389	0.490	0.588	0.676	0.764	0.845	0.916	1.000



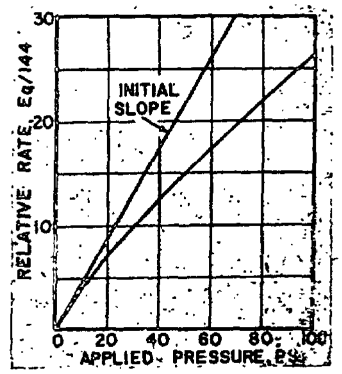


Fig. 11. Left, Variation of pressure in a filter cake.

Fig. 12. Above, Rate of flow vs. applied pressure.

Dividing (18) by (19) gives

$$\frac{x}{L} = \frac{\int_{P_x}^{P} y dP_x}{\int_{P_1}^{P} y dP_x} \qquad (20)$$

where $y = \epsilon^3/(1-\epsilon)^2$. Equation (20) states that the value of P_x for a given applied pressure P depends only on x/L and not on q, μ, or S_o.

In order to solve Equations (18)-(20) it is necessary to relate ϵ to P_x as was previously done in Figure 10. With this information it is possible to obtain the value of the Kozeny function $\epsilon^3/(1-\epsilon)^2$ for each P_x and numerically to integrate Equations (18)-(20).

EXAMPLE 1. Suppose a liquid is to flow through a fixed bed of the kaolin for which ϵ vs. p_s is plotted in Figure 10. Calculations for obtaining the p_s vs. x/L relationship at 100 lb./sq.in. are shown in Table 3.

A plot of the data in Table 3 for an applied pressure of 100 lb./sq.in. is shown in Figure 11 along with curves for 25, 50, and 75 lb./sq.in. While the curves do not depart greatly from straight lines, there is definite curvature. On the 100 lb./sq.in. curve, the slope at $x/L = 1.0$ is about 2.4 times the slope at $x/L = 0$.

Rate of Flow vs. Applied Pressure

Of more interest than the pressure drop throughout a fixed bed is the variation of the rate of flow with the applied pressure. Consider a rectangular solid of constant thickness L in which the liquid flows in one surface and out the opposite. Suppose the total mass of dry solid is w lb./sq.ft. If ϵ represents the per cent voids at distance x below the surface and the true density is ρ_s, then the mass of solids per square foot in distance dx is given by

$$dw = (\text{density})(\text{volume of solids})$$

$$= \rho_s(1-\epsilon)dx \qquad (21)$$

where $(1-\epsilon)dx$ represents the true volume of the solids. Integrating from zero to L yields

$$w = \rho_s \int_0^L (1-\epsilon)dx \qquad (22)$$

The variable of integration in (22) can be changed from x to P_s with the aid of the Kozeny equation (16). The differential dx is obtained from (16) and substituted into (22). The limits of integration are changed from L to P_1 and O to P. Thus combining (16) and (22) and changing the limits of integration to avoid a negative sign leads to

$$q = \frac{\rho_s g_c}{k \mu w S_o^2} \int_{P_1}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s \qquad (23)$$

The value of this integral can be found graphically in a manner similar to that used in Example 1. A plot of a relative rate $k \mu w S_o A/\rho_s g_c = qE$ vs. P for the solid of Example 1 is shown in Figure 12. It can be seen that the decrease in the porosity with increasing pressure has a pronounced effect on the flow rate. Calculations show that the average resistance defined as P/q at 100 lb./sq.in. is 1.5 times the resistance at 5 lb./sq.in.

Two Solids in Series

If the liquid is flowing through two solids in series as would occur with two layers of graded sand or a precoat in filtration, Equation (23) must be modified. Suppose that the pressure drops from P to P_1 in the first solid and from P_1 to P_2 in the second solid, where P_2 is now the pressure required to overcome the septum resistance. Equations similar to (23) can be set up for each solid subject to the condition that the flow rate is the same through each. Thus

$$q = \frac{g_c \rho_1}{k\mu w_1 S_1^2} \int_{P_1}^{P} \frac{\epsilon_1^3}{(1-\epsilon_1)} dP_s$$

$$= \frac{g_c \rho_2}{k\mu w_2 S_2^2} \int_{P_2}^{P_1} \frac{\epsilon_2^3}{(1-\epsilon_2)} dP_s \quad (24)$$

where the subscripts 1 and 2 are used to denote the values of the parameters ϵ, ρ, w, and S_o for the two solids. Theoretically, P_1 can be eliminated from these two equations to give q as a function of the inlet and outlet pressures P and P_2. If both solids were incompressible and ϵ_1 and ϵ_2 were constants, integration of (24) would produce

$$\frac{k\mu q}{g_c} = \frac{\rho_1}{w_1 S_1^2} \frac{\epsilon_1^3}{(1-\epsilon_1)} (P - P_1)$$

$$= \frac{\rho_2}{w_2 S_2^2} \frac{\epsilon_2^3}{(1-\epsilon_2)} (P_1 - P_2) \quad (25)$$

Elimination of P_1 yields

$$q = \frac{g_c(P - P_2)}{(w_1 \alpha_1 + w_2 \alpha_2)} \quad (26)$$

where

$$\alpha_1 = \frac{k(1-\epsilon_1)S_1^2}{\rho_1 \epsilon_1^3} \quad (27)$$

and

$$\alpha_2 = \frac{k(1-\epsilon_2)S_2^2}{\rho_2 \epsilon_2^3} \quad (28)$$

The quantities α_1 and α_2 are the resistances per pound mass per square foot of solid. If the porosities are not constant, it is necessary to perform the integrals in (24) by numerical methods and adjust the value of P_1 so that the equality is realized.

Constant Rate Filtration

In the previous sections the analysis has been simplified in that the thickness and the mass of cake have not changed with time. In a filtration process, the thickness changes with time; and the equations must be modified. Fundamentally, it is desired to obtain relationships between the volume filtered, pressure, and time.

If the filtration is carried out at constant rate, the mass of material deposited per square foot w and the applied pressure P which is the upper limit of the integrals in Equations (23) and (2´) will vary together. As w increases in Equation (23) or (24), the value of P must also increase if q remains constant. If v is the volume of filtrate per square foot, m the average ratio of the mass of wet to dry cake, and s the per cent solids in the slurry, then

$$w = s\rho v/(1 - ms) = s\rho q\theta/(1 - ms) \quad (29)$$

where θ is the time and $v = q\theta$. In the case of a dilute slurry where $(1 - ms)$ is approximately unity, (29) reduces to

$$w = s\rho v = s\rho q\theta \quad (30)$$

Combining (29) with (23) gives

$$\frac{s\rho\mu}{1 - ms} \frac{kS_o^2}{\rho_s g_c} q^2\theta = \int_{P_1}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s \quad (31)$$

This equation represents the desired solution for constant rate filtration in which θ is given as a function of the upper limit P. The lower limit P_1 is constant only when the septum resistance does not change, e.g., when the solids are laid down directly on something like a wire screen. If a precoat is involved, the changing pressure P would cause the precoat resistance to increase, and P_1 would not be constant. If P_1 is small, slight changes would have little effect on the results.

If s is not small enough so that ms may be neglected, m must be obtained by an integration. Thus

$$m = \frac{\text{mass wet cake}}{\text{mass dry cake}}$$

$$= 1 + \frac{\text{mass moisture}}{\text{mass dry cake}} \quad (32)$$

$$m = 1 + \frac{\rho \int_0^L \epsilon dx}{\rho_s \int_0^L (1-\epsilon) dx} \quad (33)$$

Substituting for dx as obtained from Kozeny's equation yields

$$m = 1 + \frac{\rho \int_{P_1}^{P} \frac{\epsilon^4}{(1-\epsilon)^2} dP_s}{\rho_s \int_{P_1}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s} \quad (34)$$

When ϵ does not vary too widely, Equation (34) may be replaced by

$$m = 1 + \frac{\rho\epsilon}{\rho_s(1-\epsilon)} \quad (35)$$

which is usually sufficiently accurate if an average value of ϵ for liquid pressures between P_1 and P is used.

For the case of constant porosity, integration of (31) gives the usual simple form of the constant rate equation. Thus

$$P - P_1 = \frac{s\rho\mu}{1 - ms} \frac{\alpha}{g_c} q^2\theta \quad (36)$$

where α, the filtration resistance, is given by

$$\alpha = \frac{k(1-\epsilon)S_o^2}{\rho_s \epsilon^3}. \quad (37)$$

Equation (36) would only be valid for an incompressible solid in which the voids were independent of the pressure.

EXAMPLE 2. Suppose a solid having the same porosity characteristics as the kaolin in Example 1 is to be filtered in a unit with negligible septum resistance at a constant rate of 0.5 gal./(min.)(sq.ft.). Further suppose that the slurry is dilute and contains 0.5% solids by weight. Other parameters have the values:

μ = 1 centipoise
ρ = 62.4 lb.m./cu.ft.
ρ_s = 124.8 lb.m./cu.ft.
S_o = 2,000,000 sq.ft./cu.ft.
θ' = min. = $\theta/60$

Since s = 0.005, ms will be small, and $(1 - ms)$ may be taken as unity. Changing the rate from gallons per minute to consistent units yields a value of (0.5)/(60)(7.48) = 0.00113 cu.ft./(sec.)(sq.ft.). Substituting the values in (31) yields

$$\frac{(0.005)(62.4)(0.000672)(5)(4)(10^{12})(0.00113)^2(60)\theta'}{(124.8)(32.2)}$$

$$= 77.8\theta' = \int_0^P \frac{\epsilon^3}{(1-\epsilon)} dP_s \quad (38)$$

In (38) P is in pounds per square foot. Changing to pounds per square inch leads to

$$\theta' = 1.85 \int_0^p \frac{\epsilon^3}{(1-\epsilon)} dp_s \quad (39)$$

Table 4

p_s	0	10	20	30	40	50	60	70	80	90	100
ϵ	0.575	0.534	0.520	0.512	0.505	0.499	0.494	0.489	0.485	0.481	0.476
p_w	100	90	80	70	60	50	40	30	20	10	0
$\frac{\epsilon^3}{1-\epsilon}$	0.448	0.331	0.292	0.275	0.260	0.248	0.238	0.228	0.222	0.213	0.206

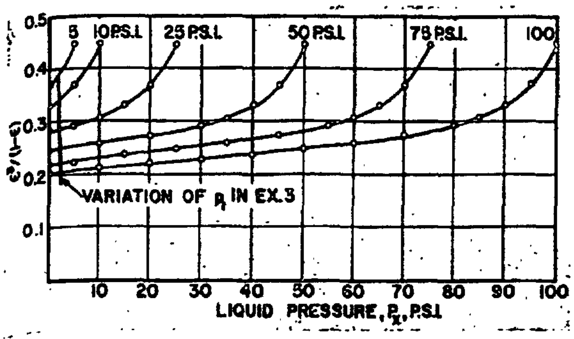


Fig. 13. Variation of the Kozeny function with liquid pressure.

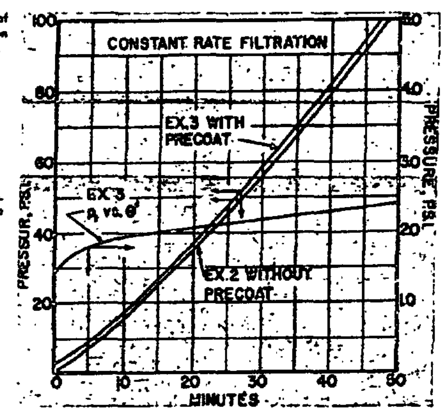


Fig. 14. Pressure vs. time in constant rate filtration.

for each applied pressure p, the integral must be evaluated numerically to give the time. In Table 4, the calculations are illustrated for p = 100 lb./sq.in.

The integral corresponding to Equation (31) is found by calculating the area under the curve of $\epsilon^3/(1-\epsilon)$ plotted against p_s. The graphs of $\epsilon^3/(1-\epsilon)$ vs. p_s for applied pressures ranging from 5-100 lb./sq.in. are demonstrated in Figure 13. The area under the curve for 100 lb./sq.in. is found to be 26.39 lb./sq.in. or 3800 lb./sq.ft. The time required to reach 100 lb./sq.in. is obtained from Equation (39) and is t = (1.85) (26.15) = 48.3 min. In Figure 14, the applied pressure is plotted against the time. The curve for Example 2 is typical of the type encountered in the constant rate filtration of a compressible material.

Constant Rate Filtration with a Precoat

Equation (24) for two fixed beds of solids in series can be modified to give the equations for constant rate filtration with a precoat. The mass of solids per square foot w_2 of the precoat will remain constant while w_1 will change with the time in accord with Equation (29). Solving for w in (29) and substituting the result in (24) in the place of w, gives

$$q = \frac{g_c(1-ms)\rho_1}{k\mu s\rho q\theta S_1^2}\int_{P_1}^{P}\frac{\epsilon_1^3}{(1-\epsilon_1)}dP_s$$

$$= \frac{g_c\rho_2}{k\mu w_2 S_2^2}\int_{P_2}^{P_1}\frac{\epsilon_2^3}{(1-\epsilon_2)}dP_s \qquad (39)$$

The value of P_2 will equal $\mu R_m q/g_c$ if the septum resistance does not change and the flow is viscous, while P_1 will increase as the filtration process continues. For incompressible solids (39) becomes on integration

$$\frac{k\mu q}{g_c} = \frac{(1-ms)\rho_1}{s\mu q\theta S_1^2}\frac{\epsilon_1^3}{(1-\epsilon_1)}(P-P_1)$$

$$= \frac{\rho_2}{w_2 S_2^2}\frac{\epsilon_2^3}{(1-\epsilon_2)}(P_1-P_2) \qquad (40)$$

In terms of α_1 and α_2 as defined by Equations (27) and (28), Equation (40) may be transformed into

$$q = \frac{1-ms}{s\rho q\theta}\frac{g_c(P-P_1)}{\alpha_1} = \frac{g_c(P_1-P_2)}{\alpha_2 w_2} \qquad (41)$$

Elimination of P_1 and substitution of $\mu R_m q$ for $g_c P_2$ yields

$$g_c P - \mu(R_m + w_2\alpha_2)q = \frac{s\rho\mu}{1-ms}\alpha_1 q^2\theta \qquad (42)$$

The solution of Equation (39) in the most general case requires the numerical determination of P_1 and θ for each applied pressure P. The determination of the exact solution is rather tedious unless the simplifying assumption is made that the porosity of the precoat is constant. If P_1 is not large, the variation of P_s and ϵ_2 through the precoat will be small. The porosity will be at its minimum value at the side where the liquid exits. As a satisfactory approximation the value of ϵ_2 may be taken as constant at its *minimum* value and can be calculated on the basis of the maximum value of P_s. At the side where the filtrate leaves the precoat and $p_x = p_2$, p_s is at a maximum and is given by

$$p_s = p - p_2 \qquad (43)$$

The value of ϵ_2 corresponding to this value of p_s may be calculated by using Equation (4)

$$\epsilon = \epsilon_0(p-p_2)^{-c} \qquad (44)$$

or it may be obtained directly from the experimental data. Since ϵ_2 is being assumed constant, the value of P_1 may be determined from Equation (41), thus

$$P_1 = P_2 + \mu w_2\frac{k(1-\epsilon_2)S_2^2}{g_c\rho_2\epsilon_2^3}q$$

$$= P_2 + \mu w_2\frac{\alpha_2}{g_c}q \qquad (45)$$

Values of ϵ_2 and α_2 will change during the course of the filtration. As the filtration pressure P builds up, the value of ϵ_2 will decrease and α_2 will increase. With the values of P and P_1 fixed, θ is found to be

$$\theta = \frac{(1-ms)}{\mu s\rho q^3}\frac{g_c\rho_1}{kS_1^2}\int_{P_1}^{P}\frac{\epsilon_1^3}{(1-\epsilon_1)}dP_s \qquad (46)$$

EXAMPLE 3. Suppose a heavy precoat of 1.0 lb./sq.ft. of a filter aid with the following characteristics is laid down upon a septum with negligible resistance:

sp.gr. = 2.3
S_0 = 1,000,000 sq.ft./cu.ft.
$\epsilon_s = 0.85\, p_s^{-}$[illegible]

Further suppose that the material of Example 3 is being filtered under the following conditions which are identical with those of the previous example:

$$\text{sp.gr.} = 2.0$$
$$S_1 = 2{,}000{,}000 \text{ sq.ft./cu.ft.}$$
$$s = 0.005$$
$$1 - ms = 1.0 \text{ (approximately)}$$
$$q = 0.5 \text{ gal./min./(sq.ft.)}$$
$$\mu = 1 \text{ centipoise}$$
$$\rho = 62.4 \text{ lb.m./cu.ft.}$$
$$p_2 = 0$$
$$\theta' = \text{min.}$$

On changing to lb./sq.in. and minutes, Equation (39) becomes

$$0.00112 = \frac{0.00103}{\theta'} \int_{p_1}^{p} \frac{\epsilon_s^3}{(1-\epsilon_s)} dp_s$$

$$= 0.000197 \frac{\epsilon_1^3}{(1-\epsilon_1)} p_1 \qquad (47)$$

where p_1 is in pounds per square inch and ϵ_1 is the value of the porosity corresponding to the solid pressure at the liquid exit from the precoat. Equations (45) and (46) become

$$P_1 = \frac{(0.000672)(5)(10^{13})(0.00112)}{(32.2)(2.3)(62.4)(144)} \frac{1-\epsilon_1}{\epsilon_1^3}$$

$$= 5.66 \frac{1-\epsilon_1}{\epsilon_1^3} \qquad (48)$$

$$\theta' = 1.85 \int_{p_1}^{p} \frac{\epsilon_s^3}{(1-\epsilon_s)} dp_s \qquad (49)$$

The value of θ' in minutes corresponding to the applied pressure p in pounds per square inch as given by (49) can be obtained by graphical integration of $\epsilon_1^3/(1-\epsilon_1)$ between p_1 and p. Since the solid being filtered is the same as that used in Example 2, the value of $\epsilon_1^3/(1-\epsilon_1)$ will be the same for equal values of p_s and the curves of Figure 13 may be employed for obtaining the values of the integrals. In Example 2, where there was no precoat and negligible septum resistance, the integration was carried out with the lower variable of integration in (49) equal to zero. In this example, the lower limit of integration will not be zero, and the numerical value of the integral in this example will be smaller for a given applied pressure p than in the previous example. This amounts to saying that the time required to reach a given pressure will be smaller with the added resistance of the precoat.

In order to calculate the value of the integral in Equation (46), it is first necessary to obtain the values of p_1 for different values of the applied pressure p. Since p_2 is zero in this example p_s will equal p at the outlet, and ϵ_2 can be calculated from the relation $\epsilon_2 = 0.85\ p_s^{-0.016} = 0.85\ p^{-0.016}$. With ϵ_2, p_1 can be obtained from Equation (4) as illustrated in Table 5.

Examination of the first two columns in Table 5 reveals the fact that p is less than p_1. Since p is the applied pressure, it must be greater than or equal to p_1. At $p = 1.46$ lb./sq.in., both p and p_1 are equal, and the time is zero. Therefore 1.46 lb./sq.in. represents the initial pressure necessary to overcome the resistance of the precoat. In Figure 14 the variation of p_1 with time is shown. When the total pressure had increased to 100 lb./sq.in., p_1 had changed to only 2.37 lb./sq.in. Actually with a smaller and more reasonable quantity of precoat, the values of p_1 would have been smaller.

The applied pressure for the filtration of the solid with a precoat is plotted against the time in Figure 14 and can be compared with the same filtration without a precoat. Although p_1 is actually a variable, little error would have ensued from having assumed p_1 constant and equal to about 2.0 lb./sq.in.

Constant Pressure Filtration

If the applied pressure P remains constant, the rate of filtration will decrease as the cake thickness builds up. Equation (25) can be adapted to a constant pressure filtration if $q\theta$ is replaced by v and q by $dv/d\theta$. Thus

$$\frac{s\rho\mu}{(1-ms)} \frac{kS_o^2}{g_c\rho_s} v \frac{dv}{d\theta} = \frac{vdv}{Cd\theta}$$

$$= \int_{P_1}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s \qquad (50)$$

where C represents the combined value of the constants on the left-hand side of the equation. Conditions for solving (50) will differ from those of constant rate filtration in that the upper limit of the integral P will remain constant while $dv/d\theta$ and P_1 will both vary. Two cases present themselves depending upon whether the septum resistance is constant or not. In the simpler case of constant septum resistance, P_1 may be replaced by

$$g_c P_1 = \mu R_m \frac{dv}{d\theta} \qquad (51)$$

where R_m is the septum resistance and viscous flow through the septum is assumed. If ϵ is constant, on integration Equation (50) becomes

$$\frac{\alpha\rho_s\mu}{g_c(1-ms)} v \frac{dv}{d\theta} = P - P_1$$

$$= P - \frac{1}{g_c} \mu R_m \frac{dv}{d\theta} \qquad (52)$$

where α is defined by Equation (33). Integration of (52) leads to the usual parabolic relation of v to θ expressed as

$$\frac{\alpha\rho_s\mu}{g_c(1-ms)} \frac{v^2}{2} + \frac{1}{g_c} \mu R_m v = P\theta \qquad (53)$$

If ϵ is not constant, a graphical integration must be carried out to obtain the v vs. θ relationship. It is possible to solve for the volume in Equation (50) to obtain

$$v = \frac{C}{dv/d\theta} \int_{\frac{1}{g_c}\mu R_m \frac{dv}{d\theta}}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s \qquad (54)$$

Since P is fixed, the value of the integral will depend upon the lower limit of $dv/d\theta$. Thus, the integral is a function of $dv/d\theta$. It is possible to choose values of $dv/d\theta$ and then calculate v. The value of θ then can be obtained by a second numerical integration as illustrated in the following example.

EXAMPLE 4. The solution of Equation (50) will be carried out for the same solid used in Examples 2 and 3. Suppose the applied pressure is 100 lb./sq.in., the product $\mu R_m/g_c = (10{,}000)(144)$ or simply 10,000 when the pressure is in pounds per square inch and $C = 1.16(10^{-4})$. In order to solve this problem, various values of $dv/d\theta$ will be chosen, thus fixing P and the limits on (54). The initial rate of filtration is such that all the pressure drop is across

Table 5

p, applied pressure	1.0	1.2	1.46	5	10	25	50	75	100
ϵ_2	0.85	0.849	0.845	0.829	0.821	0.811	0.802	0.797	0.792
$\epsilon_1^3/(1-\epsilon_1)$	4.09	4.04	3.90	3.33	3.09	2.82	2.61	2.50	2.26
p_1, lb./sq.in.	1.38	1.40	1.46	1.70	1.83	2.01	2.17	2.26	2.37
$\int_{p_1}^{p} \frac{\epsilon_s^3}{(1-\epsilon_s)} dp_s$	.....	.....	0	1.37	3.16	7.71	14.34	20.27	25.66
θ', min.	.....	.....	0	2.54	5.85	14.3	26.5	37.5	47.5

the septum, and $p_1 = p = 100$ lb./sq.in. Thus

$$\frac{dv}{d\theta}\text{(initial)} = \frac{g_c p}{\mu R_m} = \frac{100}{10{,}000} = 0.01 \text{ ft./sec.} \tag{55}$$

The initial flow rate corresponds to 4.5 gal./min./(sq.ft.). If θ' is given in minutes and p_s in pounds per square inch (54) becomes

$$v = \frac{0.01}{dv/d\theta} \int_{10{,}000\frac{dv}{d\theta}}^{100} \frac{\epsilon^3}{1-\epsilon}\, dp_s \tag{56}$$

where $0.01 = (144)(60)(1.16)(10^{-7})$. Calculations for obtaining v vs. $dv/d\theta$ from (56) are shown in Table 6. The 100 lb./sq.in. curve for $\epsilon^3/(1-\epsilon)$ vs. p_s in Figure 13 can be used for evaluating the integral of Equation (56).

Table 6

$\frac{dv}{d\theta}$, ft./sec.	.01	.0095	.009	.0075	.005	.0025
$\frac{dv}{d\theta'}$, ft./min.	0.6	0.54	0.57	0.45	0.30	0.15
p_1	100	95	90	75	50	25
$\int_{p_1}^{100} \frac{\epsilon^3}{(1-\epsilon)} dp_s$	0	2.012	3.77	8.28	14.88	20.77
v	0	0.353	0.698	1.841	4.96	13.83
θ' (min.)	0	0.6	1.2	3.6	12.4	57.6

Plots of $d\theta'/dv$ vs. v as well as lines of constant θ' are shown in Figures 15 and 16. It can be seen in Figure 15 that the lines of $d\theta'/dv$ vs. v are virtually straight, a fact that has frequently been demonstrated experimentally. In analytical form the straight lines in Figure 15 may be represented by

$$\frac{d\theta}{dv} = \frac{2}{K} v + b \tag{57}$$

which integrates into

$$K\theta = v^2 + Kbv \tag{58}$$

Equation (58) is simply the equation of the well-known filtration parabola. Thus the graphical procedure of Example 4 for a compressible solid confirms the fact that constant pressure filtration data can be accurately represented by a parabola. There is, however, a deviation from the parabolic form at the start of the filtration where v is zero.

In Figure 16 the line for 10 lb./sq.in. is shown on an enlarged scale in order to illustrate the deviation from linearity in the $d\theta'/dv$ relationship at the beginning of the filtration. The line which is indicated as coming from Figure 15 represents the straight line through the points for times greater than 20 min. It is apparent that for the points up to 20 min. a line of smaller slope could be constructed. Since the slope is smaller at the beginning of the filtration, the resistance must increase as the filtration proceeds. An average slope up to 20 min. is 25.5 min./(ft.)^2; whereas, the final limiting value of the slope is 25.9. The value of 25.9 corresponds to a filtration resistance of $4.16(10^{11})$ ft./lb. mass.

The initial slope and filtration resistance cannot be calculated with certainty as can be seen from Figure 17. In this figure, the Kozeny function $\epsilon^3/(1-\epsilon)$ is plotted against the liquid pressure for an applied pressure of 10 lb./sq.in. The cross-hatched area under the curve between 6.0 and 10 lb./sq.in. represents the value of the integral in Equation (54). In this region the liquid pressure falls from 10 to 6.0 lb./sq.in. in the cake and from 6.0 to zero across the filter medium. The area under the curve cannot be determined with certainty in the portion of the curve between 8.58 and 10 lb./sq.in. since an extrapolation of the experimental data is necessary as indicated by the dotted extension of the curve. The porosity data were taken from Table 1 with an initial value of $p_s = 1.42 (p_s = 10 - 1.42 = 8.58$ lb./sq. in.) and an initial porosity of 0.574 corresponding to $\epsilon^3/(1-\epsilon) = 0.444$. Extrapolation to $p_s = 0$ leads to a value of $\epsilon^3/(1-\epsilon) = 0.508$ for which $\epsilon = 0.592$. In this region of low compressive pressure, the solids may be quite plastic; and it is difficult to obtain the porosity-pressure relations with accuracy. Therefore, *there is an initial region of uncertainty* in evaluation of the volume-time relationship. As time progresses, the relative effect of this region becomes smaller. The double-hatched region from 8.58 to 10 lb./sq.in. has an area of 0.045 while the total area from zero to 10 lb./sq.in. is 3.86 lb./sq.in. Thus the uncertainty becomes quite small as the volume of filtrate increases.

The initial filtration resistance can be calculated from Equation (37) using the initial value of $\epsilon^3/(1-\epsilon)$, thus

$$\alpha = \frac{k(1-\epsilon)S_o^2}{\rho_s \epsilon^3} = \frac{(5)(4)(10^{12})}{(124.8)(0.508)}$$
$$= 3.16(10^{11}) \text{ft./lb.m.} \tag{59}$$

The average filtration resistance at any time can be written as

$$\alpha_p = \frac{kS_o^2}{\rho_s} \frac{\left(P - \frac{1}{g_c}\mu R_m dv/d\theta\right)}{\int_{\frac{1}{g_c}\mu R_m \frac{dv}{d\theta}}^{P} \frac{\epsilon^3}{(1-\epsilon)} dP_s} \tag{60}$$

The ultimate resistance approached as $dv/d\theta$ becomes zero is

$$\alpha_p = \frac{kS_o^2}{\rho_s} \frac{P}{\int_0^P \frac{\epsilon^3}{(1-\epsilon)} dP_s} \tag{61}$$

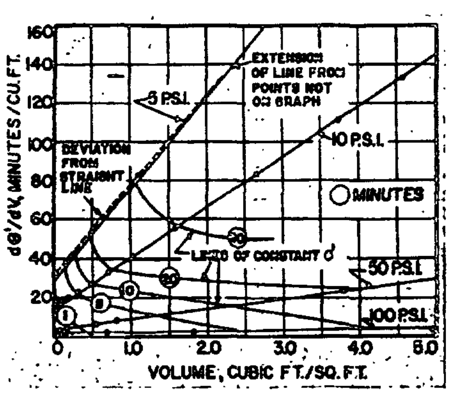


Fig. 15. $d\theta/dV$ vs. V in constant pressure filtration.

For the 10 lb./sq.in. run just considered, Equation (61) becomes

$$\alpha_P = \frac{(5)(4)(10^{12})(10)}{(124.8)(3.86)}$$

$$= 4.16\ (10^{11})\ \text{ft./lb. mass} \qquad (62)$$

This is equal to the value previously found from drawing a tangent through the $d\theta'/dv$ vs. v line. From these calculations it can be seen that the filtration resistance increases approximately 30% above its initial value. Nevertheless, the ultimate value is approached quite rapidly, the average resistance at 20 minutes being within 1.5% of the maximum value. It should be recalled that in the initial stage of the filtration, equilibrium consolidation may not be reached. Since the porosity decreases with time, the initial porosities would be larger than the equilibrium values as reported in Table 1, thus leading to smaller initial resistances than those which have been calculated. From this standpoint, the filtration resistance would be expected to increase more than 30% above its initial value.

Constant Pressure Filtration with Negligible Septum Resistance

The previous method may be simplified somewhat if R_m is taken as zero. If R_m is zero, the value of P_1 representing the lower limit of Equations (50), (54), and (56) will be zero. Thus Equation (50) becomes

$$v\frac{dv}{d\theta} = \frac{1-ms}{s\rho\mu}\,\frac{g_c\rho_s}{kS_o^2}\int_0^P \frac{\epsilon^3}{(1-\epsilon)}\,dP_s \qquad (63)$$

The right-hand side of Equation (63) is a constant for a given applied pressure P. Thus integration of (63) yields an equation of the form

$$v^2 = K\theta \qquad (64)$$

where the K is identical with the constant in Equation (58). From (64) it can be seen that v and θ are related by a perfect parabola in contrast to the previous case with a finite septum resistance. Again (64) is subject to the limitation of the time effect on consolidation. The filtration resistance found from (63) is not only constant but is also equal to the limiting value given in Equation (61). Thus, as the filter time is increased, the resistance of the medium has a lessening effect on the filtration resistance.

Constant Pressure Filtration with Precoat

The equation for constant rate filtration with a precoat can easily be modified for a constant pressure filtration. In Equation (39), q is replaced by $dv/d\theta$ and $q\theta$ by v, thus

$$\frac{k\mu}{g_c}\frac{dv}{d\theta} = \frac{(1-ms)}{s\rho v}\,\frac{\rho_1}{S_1^2}\int_{P_1}^{P} \frac{\epsilon_1^3}{(1-\epsilon_1)}\,dP_s$$

$$= \frac{\rho_2}{w_2S_2^2}\int_{P_2}^{P_1} \frac{\epsilon_2^3}{(1-\epsilon_2)}\,dP_s \qquad (65)$$

With the applied pressure P fixed and $g_cP_2 = \mu R_m dv/d\theta$, the problem rests in finding values of P_1 for various values of v. At the start of the filtration process, the value of P_1 is equal to P, and the maximum pressure is applied to the precoat. Initially then

$$\frac{k\mu}{g_c}\frac{dv}{d\theta} = \frac{\rho_2}{w_2S_2^2}\int_{\frac{1}{g_c}\mu R_m \frac{dv}{d\theta}}^{P} \frac{\epsilon_2^3}{(1-\epsilon_2)}\,dP_s \qquad (66)$$

To start the solution, a value of $dv/d\theta$ must be found which will satisfy this equality. Smaller values of $dv/d\theta$ are chosen, thereby fixing the left-hand side of (65) and the value of P_2. In the next step P_1 which replaces P in (66) is determined. With P_1 known, the integral of the center member of Equation (65) may be calculated along with values of v. After v has been found as a function of $dv/d\theta$, the values of θ may be determined by graphical integration.

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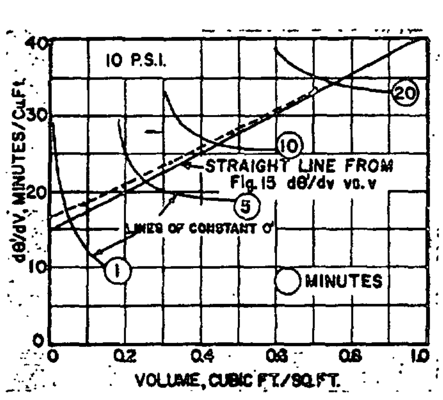

Fig. 16. Above, Enlargement of Fig. 15.

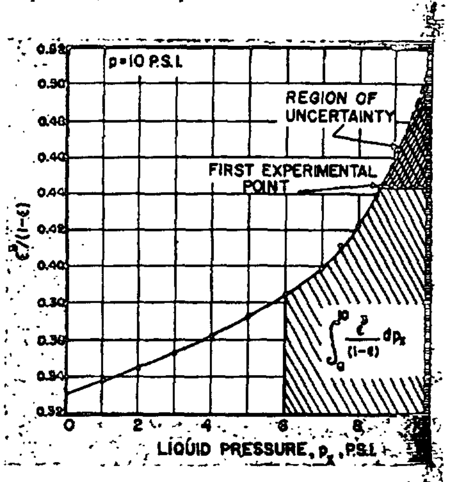

Fig. 17. Right, Integral of Kozeny function in constant pressure filtration.

Notation

A = total area perpendicular to flow, sq.ft.

b = intercept on the graph of $d\theta/dv$ vs. v, reciprocal of the initial rate of filtration, sec. or min./ft.

c = exponent occurring in Equation (4)

C = constant defined by Equation (50)

C_1, C_2 = constants in Equation (5)

D = particle diameter, ft.

E = defined by Equation (23), $k\mu wS_0^2/\rho_s g_c$, cu.ft./(sec.)(lb.f.)

g_c = conversion factor from lb.f. to lbl., numerical equivalent of the standard acceleration of gravity, 32.2 lbl./lb.f.

k = proportionality constant in the Kozeny eq., approximately equal to 5.0

K = twice the reciprocal of the slope of $d\theta/dv$ vs. v, sec. or min./sq.ft.

L = cake thickness, ft.

m = ratio of mass of wet to mass of dry cake

P = applied pressure, lb.f./sq.in.

p = applied presure, lb.f./sq.in.

P_1, P_2 = pressure at liquid exit from solid, lb.f./sq.ft.

p_1, p_2 = pressure at liquid exit from solid, lb.f./sq.in.

P_L = pressure on liquid, lb.f./sq.ft.

p_L = pressure on liquid, lb.f./sq.in.

P_s = pressure on solids, lb.f./sq.ft.

p_s = pressure on solids, lb.f./sq.in.

q = rate of filtration, cu.ft./(sq.ft.)(sec.) or superficial velocity ft./sec.

R_m = resistance of the septum, 1/ft.

R_H = hydraulic radius, ft.

s = per cent solids in slurry

S, S_0, S_v = surface area/cu.ft. of solid material, sq.ft./cu.ft.

u = average velocity of filtrate in voids of filter cake, ft./sec.

v = volume of filtrate/sq.ft., cu.ft./sq.ft.

w, w_1, w_2 = total mass of cake, lb.m./sq.ft.

x = distance measured from filtrate entrance to cake, ft.

y = value of $\epsilon^3/(1-\epsilon)^2$

z = per cent moisture

$\alpha, \alpha_{av}, \alpha_x$ = filtration resistances, ft./lb.m.

α_p = filtration resistance during constant pressure filtration, ft./lb.m.

$\epsilon, \epsilon_{av}, \epsilon_x$ = per cent voids or porosity

ϵ_d = dead volume

ϵ_0 = per cent voids at one pound per square inch, constant in Eq. (4)

ρ = density of liquid, lb./cu.ft.

ρ_s, ρ_1, ρ_2 = true density of solids, lb./cu.ft.

μ = viscosity, lb.m./(ft.)(sec.)

θ = time of filtration, sec.

θ' = time of filtration, min.

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Discussion

W. H. Elliott, (National Lead Co., Fairhaven, N. J.): In using the Kozeny equation we assume that we have streamlined flow. Is it possible ever to form a bed of solids or a filter cake in which the flow of fluid through the cake is not streamlined and the equation is not applicable?

F. M. Tiller: That is quite possible. In the equations which I showed, I factored out the viscosity term as $1/\mu$, indicating that the velocity was inversely proportional to the viscosity; if we had turbulent flow, the viscosity would be much less important in calculating the resistance which would be proportional to about 0.2 power of the viscosity. We would find also that the velocity would be proportional to the square root of the pressure drop. There is no reason that we couldn't modify the equations for turbulent flow.

Anonymous: Have you ever found any data showing application of the Kozeny equation for highly viscous materials passing through porous beds—materials having a viscosity of 1,000 poises with high pressure drop?

F. M. Tiller: That represents quite a difficult problem. In the first place, many of these liquids with so high a viscosity, particularly like Du Pont viscose, may behave as non-Newtonian liquids. The laws governing the flow rates will not follow the ordinary Kozeny equation. That is a problem which I don't think has been completely solved. There are many other problems which are side products of the filtration of highly viscous materials. It is difficult to tell when the filtrate has been properly clarified and to evaluate filter aid characteristics.

Anonymous: Would you expect sand to compress under pressures of possibly 3,000 or 5,000 lb./sq.in.?

F. M. Tiller: Yes, you certainly would. In fact, you'd probably have crushing of the particles with pressures of that magnitude. Most of the work on sands, clays, and similar materials, has been done by men in the soil mechanics field. As to whether sand is compressible or not depends largely upon particle size. Many fine sands are compressible. In fact, the differentiation between clay and sand is primarily based on particle size.

W. O. Webber (Humble Oil & Refining Co., Baytown, Tex.): In these derivations porosity is assumed to be a single valued function of pressure; that is, the time required to reach equilibrium is assumed to be quite short. In other words, one can see that even with a cake, possessing a significant variation of porosity with pressure, as the cake is laying itself down on the bed and approaching its equilibrium porosity, an amount of time would be required that is significant with respect to the amount of filtrate flowing. Presumably this factor would increase the effective porosity of the cake. Have you considered time lag and eliminated it as being an insignificant item?

F. M. Tiller: No, we certainly haven't eliminated it as an insignificant item. You'll find that there is a general assumption here that the equilibrium porosity is obtained instantaneously, and that is not true. Figure 4 shows the variation of porosity with time. For materials like calcium carbonate, filter aids, etc., you'll find it takes about ten minutes to reach an initial equilibrium after which a gradual creep continues for quite some time. With materials like natural clays, it may take 24 hr. to reach the initial equilibrium with the creep going on continually. Actually, the equations presented at the present time, represent the first approximation, and we can hope that the next set of equations will include the time effect. There are not many data on the time effect. It is important practically inasmuch as the porosity is at a maximum when the bed is laid down and the resistance is at a minimum. In the initial states, you may have a smaller resistance than you will have after the equilibrium porosity has been reached.

Anonymous: In the operation of a Schriver plate and frame-type press or a Sweetland leaf-type press, would your remarks about controlling the pressure of the liquid indicate that longer operation of the filter without cleaning the leaves or plate or more rapid filtration could be obtained by keeping a relatively high back pressure on the liquid leaving the unit?

F. M. Tiller: The compressibility of the solids is dependent upon the pressure drop of the liquid through the solid. If there is point contact between particles, it will not make any difference as far as the compressive pressures are concerned, whether there is a pressure drop from 10 lb. to 0 lb. or 100 lb. to 90 lb. through the cake. The resistance should essentially be the same.

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